Bayesian Semiparametric Models for Spatially Correlated and Arbitrarily Censored Data

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(Joint work with Timothy Hanson, University of South Carolina)
1 Motivation

2 Bayesian Semiparametric Models

3 Data Analysis
   - Childhood Mortality Data
   - Loblolly Pine Trees Data

4 Summary
Outline

1. Motivation
2. Bayesian Semiparametric Models
3. Data Analysis
4. Summary
Spatially Correlated Survival Data

- Spatial survival data are commonly observed in diverse areas such as epidemiology, environmental health, ecology, etc.
- Data structure: \( \{(t_{ij}, x_{ij}, s_i) : i = 1, \ldots, m; j = 1, \ldots, n_i\} \), where
  - \( t_{ij} \) is a random survival time for individual \( j \) within region/location \( s_i \),
  - \( x_{ij} \) is a related \( p \)-vector of covariates, and
  - \( \{s_i\}_{i=1}^m \) is a set of distinct regions/locations.
- Spatial survival data are often classified into two types:
  - georeferenced data, where \( s_i \) is recorded as longitude and latitude;
  - areal data, where \( s_i \) represents a geographic region, e.g. county, state.
Arbitrary Censoring

- The survival time $t_{ij}$ is said to be *arbitrarily censored* if we only observe an interval $(a_{ij}, b_{ij})$ in which $t_{ij}$ lies, where $0 \leq a_{ij} \leq b_{ij} \leq \infty$.

- The arbitrary censoring is a mixture of
  - right censoring with $b_{ij} = \infty$,
  - left censoring with $a_{ij} = 0$,
  - interval censoring with $0 < a_{ij} < b_{ij} < \infty$,
  - and noncensoring with $a_{ij} = b_{ij}$, i.e., we define $(x, x) = \{x\}$.

- The observed data is $\{(a_{ij}, b_{ij}, x_{ij}, s_i) : i = 1, \ldots, m; j = 1, \ldots, n_i\}$.

- The goal is to model $S_{x_{ij}}(t) = P(t_{ij} > t|x_{ij})$ semiparametrically in the presence of arbitrary censoring and spatial dependence.
Popular Semiparametric Models

- Three commonly used models:
  - proportional hazards (PH) model
    \[ S_{x_{ij}}(t) = S_0(t)e^{x'_{ij}\beta + v_i} \]
  - accelerated failure time (AFT) model
    \[ S_{x_{ij}}(t) = S_0(e^{x'_{ij}\beta + v_i}t) \]
  - proportional odds (PO) model
    \[ \frac{S_{x_{ij}}(t)}{1 - S_{x_{ij}}(t)} = e^{-x'_{ij}\beta - v_i} \frac{S_0(t)}{1 - S_0(t)}. \]

- Here \( v_i \) is an unobserved frailty associated with \( s_i \), and \( S_0(t) \) is the baseline survival function corresponding to \( x_{ij} = 0 \) and \( v_i = 0 \).

- \( e^{x'_{ij}\beta} \) can be interpreted as the relative risk under PH, acceleration factor under AFT, and odds factor under PO of subject \( x_{ij} \) relative to \( x_{ij} = 0 \).
Zhang and Davidian (2008, Biometrics) model the baseline $f_0(t)$ by a polynomial-based seminonparametric density estimator under all three models for arbitrarily censored data, but not for spatial data.

Zhao, Hanson and Carlin (2009, Biometrika) consider a mixture of Polya trees prior on $f_0(t)$ under all three models for right censored areal data. The mixing is not very good under AFT.

Pan et al. (2014, CSDA), Lin et al. (2015, LiDA) and Wang et al. (2016, Biometrics) use monotone splines to approximate the baseline hazard $H_0(t)$ under PH for interval censored data. With appropriate data argumentations, the parameter estimates can be found via simple Gibbs sampler or EM algorithm. But their method has not been extended to fit the AFT model.
Available Packages

- **BayesX** (Belitz et al. 2015) uses penalized B-splines to model log baseline hazard under the PH. It allows for arbitrary censoring and spatial frailties (for both georeferenced and areal data).

- **ICBayes** (Pan et al. 2014) can be used to fit the PH and PO for interval-censored data, but not for spatial data yet.

- **bayesSurv** (Komárek and Lessffre, 2007) fits the AFT based on finite mixtures of normal and approximating B-splines.

- However, there is no package that can fit all three models using the same treatment on the baseline function, and allowing for arbitrary censoring and spatial dependence simultaneously.
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Consider a Bernstein polynomial (BP) prior (Petrone 1999),

\[ b(x|J, w_J) = \sum_{j=1}^{J} w_{Jj}\beta(x|j, J - j + 1), \]

where \( w_J = (w_{J1}, \ldots, w_{JJ})' \sim \text{Dirichlet}(\alpha, \ldots, \alpha) \) and \( \beta(\cdot|a, b) \) is the density of \( \text{Beta}(a, b) \).

Under mild conditions, for any density \( f \) with support \((0, 1)\),

\[ \sup_{0<x<1} |f(x) - b(x|J, w_J)| = O(J^{-1}). \]

The corresponding cumulative distribution function (cdf) is

\[ B(x|J, w_J) = \sum_{j=1}^{J} w_{Jj}I_x(j, J - j + 1), \]

where \( I_x(a, b) \) is the cdf associated with \( \beta(x|a, b) \).

Note \( E[b(x|J, w_J)] = 1 \) and \( E[B(x|J, w_J)] = x \) for \( x \in (0, 1) \).
Bernstein Polynomial Prior on $S_0(t)$

Let $\{S_\theta : \theta \in \Theta\}$ denote a parametric family of survival functions with support on $\mathbb{R}^+$, e.g., log-logistic, lognormal, or Weibull.

Note $S_\theta(t)$ always lies in the interval $(0, 1)$ for $0 < t < \infty$, so for a relatively large $J$, $S_0(t)$ and $f_0(t)$ can be well approximated by

$$S_0(t|J, w_J, \theta) = B(S_\theta(t)|J, w_J),$$

and

$$f_0(t|J, w_J, \theta) = b(S_\theta(t)|J, w_J)f_\theta(t),$$

where $f_\theta$ is the density associated with $S_\theta$.

Clearly, $E[S_0(t|J, w_J, \theta)] = S_\theta(t)$ and $E[f_0(t|J, w_J, \theta)] = f_\theta(t)$.

The weights $w_J$ “adjust” the shape of $S_0$ relative to $S_\theta$. 

Bernstein Polynomial Prior with $J = 15$ and $\alpha = 0.5$
Bernstein Polynomial Prior with $J = 15$ and $\alpha = 5$
Bernstein Polynomial Prior with $J = 15$ and $\alpha = 100$
Prior on Frailties $\mathbf{v} = (v_1, \ldots, v_m)'$

- Areal data: conditionally autoregressive (CAR)
  - Let $e_{ij} = 1$ if $i$ and $j$ are adjacent and $e_{ij} = 0$ otherwise; set $e_{ii} = 0$.
  - The ICAR prior is defined through a set of conditional distributions
    $$v_i | \{v_j\}_{j \neq i} \sim N \left( \sum_{\{j: j \neq i\}} e_{ij} v_j / e_{i+}, \frac{\tau^2}{e_{i+}} \right), \ i = 1, \ldots, m,$$
    where $e_{i+} = \sum_{\{j: j \neq i\}} e_{ij}$. 

- Georeferenced data: Gaussian random field (GRF)
  - Assume $v \sim N_m(0, \tau^2 \mathbf{R})$, where $\mathbf{R}[i, j] = e^{-\phi \|s_i - s_j\|} \nu$.
    Here $\phi > 0$ measures the spatial decay over distance, and $\nu \in (0, 2]$ is pre-specified.
  - The GRF prior is also a set of conditional distributions
Prior on Frailties $\mathbf{v} = (v_1, \ldots, v_m)'$

- **Areal data:** conditionally autoregressive (CAR)
  - Let $e_{ij} = 1$ if $i$ and $j$ are adjacent and $e_{ij} = 0$ otherwise; set $e_{ii} = 0$.
  - The ICAR prior is defined through a set of conditional distributions
    
    $$v_i | \{v_j\}_{j \neq i} \sim N \left( \sum_{\{j: j \neq i\}} e_{ij} v_j / e_{i+}, \tau^2 / e_{i+} \right), \quad i = 1, \ldots, m,$$

    where $e_{i+} = \sum_{\{j: j \neq i\}} e_{ij}$.

- **Georeferenced data:** Gaussian random field (GRF)
  - Assume $\mathbf{v} \sim N_m(\mathbf{0}, \tau^2 \mathbf{R})$, where $\mathbf{R}[i, j] = e^{-(\phi \| \mathbf{s}_i - \mathbf{s}_j \|^\nu)}$. Here $\phi > 0$ measures the spatial decay over distance, and $\nu \in (0, 2]$ is pre-specified.
  - The GRF prior is also a set of conditional distributions
    
    $$v_i | \{v_j\}_{j \neq i} \sim N \left( - \sum_{\{j: j \neq i\}} p_{ij} v_j / p_{ii}, \tau^2 / p_{ii} \right), \quad i = 1, \ldots, m,$$

    where $p_{ij} = (\mathbf{R}^{-1})[i, j]$. 
The Likelihood

- Observed data \( \{(a_{ij}, b_{ij}, x_{ij}, s_i) : i = 1, \ldots, m; j = 1, \ldots, n_i\} \).

- The likelihood for \((w, \theta, \beta, v)\) is given by

\[
L(w, \theta, \beta, v) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} \left[ S_{x_{ij}}(a_{ij}) - S_{x_{ij}}(b_{ij}) \right]^{I\{a_{ij} < b_{ij}\}} f_{x_{ij}}(a_{ij})^{I\{a_{ij} = b_{ij}\}},
\]

where \(f_{x_{ij}}\) is the density associated with \(S_{x_{ij}}\).

- The posterior density given the data \(D\) is

\[
p(w, \theta, \beta, v|D) \propto L(w, \theta, \beta, v)p(w|\alpha)p(\alpha)p(\theta)p(\beta)p(v|\tau^2, \phi)p(\tau^2)p(\phi)
\]

where each \(p(\cdot)\) represents a prior density, and \(p(\phi)\) is needed only for georeferenced data.
Prior Specifications

- Assume \( \alpha \sim \Gamma(a_0, b_0) \), \( \theta \sim N_2(\theta_0, V_0) \), \( \beta \sim N_p(\beta_0, S_0) \), \( \tau^{-2} \sim \Gamma(a_{\tau}, b_{\tau}) \), and \( \phi \sim \Gamma(a_{\phi}, b_{\phi}) \).

- Note that when \( w_{jj} = 1/J \) the underlying parametric model with \( S_0(t) = S_\theta(t) \) is obtained and \( L(w, \theta, \beta, v) \) is equal to the corresponding parametric likelihood function.

- A fit from a standard parametric survival model can provide good starting values and proposals for MCMC.

- Default hyperprior values: \( a_0 = b_0 = 1 \), \( a_{\tau} = b_{\tau} = 0.001 \), \( \beta_0 = 0 \), \( S_0 = 10^{10} I_p \), \( \theta_0 = \hat{\theta} \), and \( V_0 = 10 \hat{V} \), where \( \hat{\theta} \) is the parametric point estimates of \( \theta \) and \( \hat{V} \) is its estimated covariance.

- For georeferenced data, set \( a_{\phi} = 1 \) and choose \( b_{\phi} \) so that \( \Pr(\phi > \phi_0) = 0.95 \), where \( \phi_0 \) satisfies \( e^{-(\phi_0 \cdot \max \|s_i - s_j\|)\nu} = 0.001 \).
MCMC Overview

- Set $\mathbf{z}_{J-1} = (z_1, \ldots, z_{J-1})'$ with $z_j = \log(w_j) - \log(w_J)$.

- The $\beta$, $\theta$, $\mathbf{z}_{J-1}$, $\alpha$ and $\phi$ are all updated using adaptive Metropolis samplers (Haario et al., 2001, Bernoulli), where the initial proposal variance is from the underlying parametric fit for $\beta$ and $\theta$, is 0.16$I_{J-1}$ for $\mathbf{z}_{J-1}$, and 0.16 for $\alpha$ and $\phi$.

- The frailty term $v_i$ is updated individually via Metropolis-Hastings, where the proposal is the conditional prior variance of $v_i|\{v_j\}_{j \neq i}$.

- The $\tau^{-2}$ is updated via Gibbs sampler from its full conditional.

- For large $m$, we use the full scale approximation (FSA) (Sang and Huang, 2012, JRSSB) to inverse $\mathbf{R}_{m \times m}$. 
Multiply $\beta_k$ by a latent indicator $\gamma_k$, where $\gamma_k = 1$ ($\gamma_k = 0$) indicates presence (absence) of covariate $x_k$ in the model, $k = 1, \ldots, p$.

Consider the prior

$$
\beta \sim N_p(0, gn(X'X)^{-1}), \quad \gamma_k \, iid \sim Bern(0.5),
$$

where $X = (x_1, \ldots, x_n)'$ is the usual design matrix with mean-centred covariates, i.e. $1'X = 0'_n$.

Hanson, Branscum and Johnson (2014) note that it is reasonable to assume $e^{x'_i\beta} \sim \log N(0, gp)$ in many settings.

The constant $g$ is chosen so that $\Pr(e^{x'_i\beta} < 10) = .9$. It follows that $g = 3.228/p$. 
The survival time $t_{ij}$ may be left-truncated at $u_{ij} \geq 0$, if $u_{ij}$ is the time when subject $ij$ is first observed.

Given the observed left-truncated data $\{(u_{ij}, a_{ij}, b_{ij}, x_{ij}, s_i)\}$, the likelihood function becomes

$$L(w, \theta, \beta, v) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} [S_{x_{ij}}(a_{ij}) - S_{x_{ij}}(b_{ij})] I\{a_{ij} < b_{ij}\} f_{x_{ij}}(a_{ij}) I\{a_{ij} = b_{ij}\} / S_{x_{ij}}(u_{ij}).$$
The survival time $t_{ij}$ may be left-truncated at $u_{ij} \geq 0$, if $u_{ij}$ is the time when subject $ij$ is first observed.

Given the observed left-truncated data $\{(u_{ij}, a_{ij}, b_{ij}, x_{ij}, s_i)\}$, the likelihood function becomes

$$L(w, \theta, \beta, v) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} [S_{x_{ij}}(a_{ij}) - S_{x_{ij}}(b_{ij})]^{I\{a_{ij} < b_{ij}\}} f_{x_{ij}}(a_{ij})^{I\{a_{ij} = b_{ij}\}} / S_{x_{ij}}(u_{ij}).$$

Assume $x_{ij}(t)$ is a step function:

$$x_{ij}(t) = \sum_{k=1}^{o_{ij}} x_{ij,k} I(t_{ij,k} \leq t < t_{ij,k+1}), \text{ where } t_{ij,1} = u_{ij}, t_{ij,o_{ij}+1} = \infty.$$

Replace the observation $(u_{ij}, a_{ij}, b_{ij}, x_{ij}(t), s_i)$ by new $o_{ij}$ observations $(t_{ij,1}, t_{ij,2}, \infty, x_{ij,1}, s_i), (t_{ij,2}, t_{ij,3}, \infty, x_{ij,2}, s_i), \ldots, (t_{ij,o_{ij}}, a_{ij}, b_{ij}, x_{ij,o_{ij}}, s_i)$, yielding a new left truncated data set of size $N = \sum_{i=1}^{m} \sum_{j=1}^{n_i} o_{ij}$. 
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4 Summary
Application to Childhood Mortality Data in Nigeria

- Data are from the 2003 Nigeria Demographic and Health Survey.
- The state of residence is available for each child, so the data type is **areal**. There are 37 states, and the sample size is \( n = 4,363 \).
- The survival time is **age at death** of the child. It was reported in days if it was less than one month, in months if it was less than two years and otherwise in years. If the child was still alive by the date of interview, the right censoring time can be calculated in days.
- To incorporate the inconsistency of time units, we treat all survival times recorded in months or years as interval censored (details in Appendix), yielding **arbitrarily censored data**.
- Kneib (2006, CSDA) fit a proportional hazards model with CAR frailties.
### Application to Childhood Mortality Data in Nigeria

<table>
<thead>
<tr>
<th>Continuous variables</th>
<th>Mean</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age at birth (yr.)</td>
<td>28.49</td>
<td>6.48</td>
</tr>
<tr>
<td>BMI</td>
<td>22.62</td>
<td>4.21</td>
</tr>
<tr>
<td>Breastfeed duration (mo.)</td>
<td>14.48</td>
<td>7.31</td>
</tr>
<tr>
<td>Preceding interval (mo.)</td>
<td>36.46</td>
<td>21.24</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Categorical variables</th>
<th>Level</th>
<th>Proportion (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Censoring status</td>
<td>uncensored</td>
<td>1.67</td>
</tr>
<tr>
<td></td>
<td>interval censored</td>
<td>7.54</td>
</tr>
<tr>
<td></td>
<td>right censored</td>
<td>90.79</td>
</tr>
<tr>
<td>Place of delivery</td>
<td>hospital</td>
<td>32.78</td>
</tr>
<tr>
<td></td>
<td>home/other</td>
<td>67.22</td>
</tr>
<tr>
<td>Gender of child</td>
<td>male</td>
<td>49.48</td>
</tr>
<tr>
<td></td>
<td>female</td>
<td>50.52</td>
</tr>
<tr>
<td>Education</td>
<td>at least primary</td>
<td>47.26</td>
</tr>
<tr>
<td></td>
<td>no education</td>
<td>52.74</td>
</tr>
<tr>
<td>place of residence</td>
<td>urban</td>
<td>34.82</td>
</tr>
<tr>
<td></td>
<td>rural</td>
<td>65.18</td>
</tr>
</tbody>
</table>
Fit the Model using \texttt{survregbayes}

\begin{verbatim}
library(spBayesSurv);

### data preparation is omitted here ###

mcmc = list(nburn=50000, nsave=5000, nskip=9, ndisplay=1000);
res = survregbayes(formula=Surv(SurvLeft,SurvRight,type="interval2")~
                    AgeBirth+BMI+BreastfeedMonth+PrecedingInterval+
                    HospitalDelivery+Male+MotherEducation+Urban+
                    frailtyprior("car",State),data=d,survmodel="AFT",
                    mcmc=mcmc,Proximity=W,selection=FALSE);

summary(res);
\end{verbatim}

- Fit PH via \texttt{survmodel="PH"} and PO via \texttt{survmodel="PO"}.
- Set \texttt{selection=TRUE} to perform the spike and slab variable selection.
- Set \texttt{frailtyprior("grf",State)} to fit Gaussian random field frailty models and \texttt{frailtyprior("iid",State)} to fit exchangeable Gaussian frailty models.
- Remove \texttt{frailtyprior()} to fit non-frailty models.
Output of the PO Model

Posterior inference of regression coefficients
(Adaptive M-H acceptance rate: 0.18116):

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Std. Dev.</th>
<th>95%CI-Low</th>
<th>95%CI-Upp</th>
</tr>
</thead>
<tbody>
<tr>
<td>AgeBirth</td>
<td>0.013442</td>
<td>0.013473</td>
<td>0.009282</td>
<td>-0.004765</td>
<td>0.031530</td>
</tr>
<tr>
<td>BMI</td>
<td>0.005937</td>
<td>0.005889</td>
<td>0.016905</td>
<td>-0.027046</td>
<td>0.038724</td>
</tr>
<tr>
<td>BreastfeedMonth</td>
<td>-0.378559</td>
<td>-0.378286</td>
<td>0.017017</td>
<td>-0.412091</td>
<td>-0.347309</td>
</tr>
<tr>
<td>PrecedingInterval</td>
<td>-0.016541</td>
<td>-0.016465</td>
<td>0.003913</td>
<td>-0.024405</td>
<td>-0.008966</td>
</tr>
<tr>
<td>HospitalDelivery</td>
<td>-0.553409</td>
<td>-0.549641</td>
<td>0.181873</td>
<td>-0.917547</td>
<td>-0.203444</td>
</tr>
<tr>
<td>Male</td>
<td>-0.081336</td>
<td>-0.080647</td>
<td>0.120485</td>
<td>-0.316681</td>
<td>0.152651</td>
</tr>
<tr>
<td>MotherEducation</td>
<td>-0.701258</td>
<td>-0.701159</td>
<td>0.161873</td>
<td>-1.014701</td>
<td>-0.378442</td>
</tr>
<tr>
<td>Urban</td>
<td>-0.362983</td>
<td>-0.362667</td>
<td>0.148649</td>
<td>-0.661083</td>
<td>-0.075890</td>
</tr>
</tbody>
</table>

Posterior inference of conditional CAR frailty variance

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Std. Dev.</th>
<th>95%CI-Low</th>
<th>95%CI-Upp</th>
</tr>
</thead>
<tbody>
<tr>
<td>variance</td>
<td>0.7904</td>
<td>0.7117</td>
<td>0.4062</td>
<td>0.2543</td>
<td>1.7858</td>
</tr>
</tbody>
</table>

Log pseudo marginal likelihood: LPML=-2079.558
Deviance Information Criterion: DIC=4153.352
Number of subjects: n=4363
## Variable Selection

**Table**: Childhood mortality data. Selected models with high frequency.

<table>
<thead>
<tr>
<th>Model</th>
<th>Proportions</th>
<th>Selected covariates</th>
</tr>
</thead>
<tbody>
<tr>
<td>PH</td>
<td>0.402</td>
<td>Breastfeed, Preceding, Delivery, Education</td>
</tr>
<tr>
<td></td>
<td>0.138</td>
<td>Breastfeed, Preceding, Delivery, Education, Residence</td>
</tr>
<tr>
<td></td>
<td>0.124</td>
<td>Age, Breastfeed, Preceding, Delivery, Education</td>
</tr>
<tr>
<td>AFT</td>
<td>0.401</td>
<td>Breastfeed, Preceding, Delivery, Education</td>
</tr>
<tr>
<td></td>
<td>0.244</td>
<td>Breastfeed, Preceding, Delivery, Education, Residence</td>
</tr>
<tr>
<td></td>
<td>0.061</td>
<td>Age, Breastfeed, Preceding, Delivery, Education</td>
</tr>
<tr>
<td>PO</td>
<td>0.346</td>
<td>Breastfeed, Preceding, Delivery, Education, Residence</td>
</tr>
<tr>
<td></td>
<td>0.256</td>
<td>Breastfeed, Preceding, Delivery, Education</td>
</tr>
<tr>
<td></td>
<td>0.103</td>
<td>Age, Breastfeed, Preceding, Delivery, Education, Residence</td>
</tr>
</tbody>
</table>
Model Comparison and Results

Table: Model comparison.

<table>
<thead>
<tr>
<th>Model</th>
<th>Covariates</th>
<th>LPML</th>
</tr>
</thead>
<tbody>
<tr>
<td>PH</td>
<td>full</td>
<td>-2126</td>
</tr>
<tr>
<td></td>
<td>selected</td>
<td>-2125</td>
</tr>
<tr>
<td>AFT</td>
<td>full</td>
<td>-2127</td>
</tr>
<tr>
<td></td>
<td>selected</td>
<td>-2125</td>
</tr>
<tr>
<td>PO</td>
<td>full</td>
<td>-2080</td>
</tr>
<tr>
<td></td>
<td>selected</td>
<td>-2077</td>
</tr>
</tbody>
</table>

Table: Covariate effects from fitting the PO model with selected covariates.

<table>
<thead>
<tr>
<th>Covariate</th>
<th>Effect (95% CI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Breastfeed duration (mo.)</td>
<td>-0.376(-0.408, -0.347)</td>
</tr>
<tr>
<td>Preceding interval (mo.)</td>
<td>-0.015(-0.023, -0.008)</td>
</tr>
<tr>
<td>Delivery–hospital</td>
<td>-0.519(-0.876, -0.171)</td>
</tr>
<tr>
<td>Education–at least primary</td>
<td>-0.710(-1.024, -0.402)</td>
</tr>
<tr>
<td>Residence–urban</td>
<td>-0.338(-0.634, -0.047)</td>
</tr>
</tbody>
</table>
Survival Analysis of Loblolly Pine Trees

- Loblolly pine is the most commercially important timber species in Southeastern United States. Estimating its survival rate is a crucial task in forestry research.

- The dataset consists of 45,525 loblolly pine trees at 168 distinct sites, which were established in 1980-1981, and monitored annually until 2001-2002. The data type is georeferenced.

- During the 21-year follow-up, 5,379 trees experienced the death, and the rest which survived until the last follow-up are treated as right censored.

- It is of interest to investigate the association between the loblolly pine survival and several important risk factors after adjusting for spatial dependence among different sites.
Loblolly Pine Trees: Risk Factors

- **Time-independent** variables:
  - treatment (treat): 1–control, 2–light thinning, 3–heavy thinning

- **Time-dependent** variables (repeatedly measured every 3 years):
  - total height of tree in meters (TH)
  - diameter at breast height in cm (DBH)

After incorporating the time-dependent variables, the final dataset contains $N = 180,676$ observations.

Li et al. (2015, JASA) used a semiparametric PH model with several spatial frailty specifications to model the data. However, they showed that the PH assumption does not hold very well.
Loblolly Pine Trees: Risk Factors

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Loblolly Pine Trees: AFT, PH and PO

Table: Model comparison.

<table>
<thead>
<tr>
<th>Model</th>
<th>PH</th>
<th>PO</th>
<th>AFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRF frailty</td>
<td>-23,991</td>
<td>-23,882</td>
<td>-23,812</td>
</tr>
<tr>
<td>IID frailty</td>
<td>-23,966</td>
<td>-23,865</td>
<td>-23,832</td>
</tr>
<tr>
<td>Non-frailty</td>
<td>-25,508</td>
<td>-25,549</td>
<td>-25,447</td>
</tr>
</tbody>
</table>

Figure: Cox-Snell residual plot for GRF frailty PH, PO and AFT.
## Loblolly Pine Trees: GRF-AFT

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Median</th>
<th>Std. Dev.</th>
<th>95%CI-Low</th>
<th>95%CI-Upp</th>
</tr>
</thead>
<tbody>
<tr>
<td>DBH</td>
<td>-0.126270</td>
<td>-0.126519</td>
<td>0.008354</td>
<td>-0.141792</td>
<td>-0.109738</td>
</tr>
<tr>
<td>TH</td>
<td>-0.011462</td>
<td>-0.011488</td>
<td>0.001342</td>
<td>-0.014014</td>
<td>-0.008826</td>
</tr>
<tr>
<td>treat2</td>
<td>-0.388399</td>
<td>-0.387577</td>
<td>0.020644</td>
<td>-0.430511</td>
<td>-0.349127</td>
</tr>
<tr>
<td>treat3</td>
<td>-0.544378</td>
<td>-0.543409</td>
<td>0.027292</td>
<td>-0.601009</td>
<td>-0.495238</td>
</tr>
<tr>
<td>PhyReg2</td>
<td>-0.389881</td>
<td>-0.386379</td>
<td>0.106980</td>
<td>-0.593728</td>
<td>-0.200604</td>
</tr>
<tr>
<td>PhyReg3</td>
<td>-0.259512</td>
<td>-0.258088</td>
<td>0.132703</td>
<td>-0.510584</td>
<td>0.013621</td>
</tr>
<tr>
<td>C2</td>
<td>0.043812</td>
<td>0.043210</td>
<td>0.025837</td>
<td>-0.002139</td>
<td>0.097142</td>
</tr>
<tr>
<td>C3</td>
<td>0.429512</td>
<td>0.427719</td>
<td>0.031195</td>
<td>0.375179</td>
<td>0.491249</td>
</tr>
<tr>
<td>C4</td>
<td>1.101149</td>
<td>1.099480</td>
<td>0.046046</td>
<td>1.017613</td>
<td>1.194449</td>
</tr>
<tr>
<td>treat2:PhyReg2</td>
<td>0.105225</td>
<td>0.106106</td>
<td>0.031557</td>
<td>0.045876</td>
<td>0.167650</td>
</tr>
<tr>
<td>treat3:PhyReg2</td>
<td>0.246436</td>
<td>0.245954</td>
<td>0.042714</td>
<td>0.162279</td>
<td>0.331992</td>
</tr>
<tr>
<td>treat2:PhyReg3</td>
<td>-0.216354</td>
<td>-0.213024</td>
<td>0.079511</td>
<td>-0.367900</td>
<td>-0.063942</td>
</tr>
<tr>
<td>treat3:PhyReg3</td>
<td>0.125298</td>
<td>0.126770</td>
<td>0.084076</td>
<td>-0.036644</td>
<td>0.285920</td>
</tr>
</tbody>
</table>

### Variance and Range

- Variance: 0.34961, 0.34475, 0.04802, 0.26954, 0.45747
- Range: 0.2735, 0.2643, 0.0700, 0.1651, 0.4342
Survival Plots for Coastal Region under GRF-AFT
Survival Plots for Piedmont Region under GRF-AFT

- **control**
- **light thinning**
- **heavy thinning**

Survival Probability vs. Time (years)
Survival Plots for Other Region under GRF-AFT
Loblolly Pine Trees: Spatial Dependence

Under the exponential correlation \( \rho(s_i, s_j) = e^{-\phi \|s_i - s_j\|} \), the posterior mean is \( \hat{\phi} = 0.2735 \), indicating that the correlation decays by 
\( 1 - e^{-0.2735} = 24\% \) for every 1-km increase in distance.
Outline

1 Motivation

2 Bayesian Semiparametric Models

3 Data Analysis

4 Summary
Studied semiparametric AFT, PH and PO frailty models for survival data subject to arbitrary censoring and spatial dependence.

The baseline is modeled via the Bernstein polynomial prior, which is centered at a parametric family and always selects quite smooth densities, leading to more efficient posterior samplings.

Developed a function `survregbayes` within the R package `spBayesSurv` for implementing the MCMC algorithms.

Future work: marginal semiparametric models with spatial dependence modeled through copulas.
Determine the Endpoints of the Interval-Censoring

- Note the day of birth is not available for all the child who experienced the death before the interview.
- Suppose the age at death for child \( ij \) is recorded as 5 months. His birth date (month+year, but not day) is also available, say, May 1999. Then the date of death can be calculated, which is Oct 1999.
- The smallest possible survival time is the days between the last day of May 1999 and the first day of Oct 1999, denoted as \( a_{ij} \).
- The largest possible survival time is the days between the first day of May 1999 and the last day of Oct 1999, denoted as \( b_{ij} \).
- Therefore, the age at death in days for child \( ij \) lies in \((a_{ij}, b_{ij})\).
Simulation Settings

- Set $x_{ij} = (x_{i1}, x_{i2})'$ with $x_{i1} \sim \text{Bernoulli}(0.5)$ and $x_{i2} \sim \mathcal{N}(0, 1)$.
- $S_0(t) = 0.5\Phi((\log t + 1)/0.5) + 0.5\Phi((\log t - 1)/0.5)$.
- $v_i$ follows the ICAR based on the Nigeria data.
- Sample size is $n = 740$.
- The final data set yields around 20% right-censored, 40% uncensored, 25% left-censored and 15% interval-censored.
Simulation Results

Table: Simulation results based on 500 replicates for areal-referenced data. For each MCMC algorithm, 10,000 scans were thinned from 50,000 after a burn-in period of 50,000 iterations.

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter</th>
<th>BIAS</th>
<th>PSD</th>
<th>SD-Est</th>
<th>CP</th>
<th>Effective size</th>
</tr>
</thead>
<tbody>
<tr>
<td>AFT</td>
<td>$\beta_1 = 1$</td>
<td>0.003</td>
<td>0.066</td>
<td>0.062</td>
<td>0.960</td>
<td>3541</td>
</tr>
<tr>
<td></td>
<td>$\beta_2 = 1$</td>
<td>0.001</td>
<td>0.035</td>
<td>0.034</td>
<td>0.954</td>
<td>3220</td>
</tr>
<tr>
<td></td>
<td>$\tau^2 = 1$</td>
<td>0.010</td>
<td>0.296</td>
<td>0.245</td>
<td>0.968</td>
<td>7563</td>
</tr>
<tr>
<td>PH</td>
<td>$\beta_1 = 1$</td>
<td>-0.008</td>
<td>0.100</td>
<td>0.101</td>
<td>0.956</td>
<td>3299</td>
</tr>
<tr>
<td></td>
<td>$\beta_2 = 1$</td>
<td>-0.004</td>
<td>0.061</td>
<td>0.060</td>
<td>0.952</td>
<td>2025</td>
</tr>
<tr>
<td></td>
<td>$\tau^2 = 1$</td>
<td>-0.001</td>
<td>0.341</td>
<td>0.321</td>
<td>0.958</td>
<td>4862</td>
</tr>
<tr>
<td>PO</td>
<td>$\beta_1 = 1$</td>
<td>0.005</td>
<td>0.151</td>
<td>0.153</td>
<td>0.946</td>
<td>4397</td>
</tr>
<tr>
<td></td>
<td>$\beta_2 = 1$</td>
<td>0.007</td>
<td>0.083</td>
<td>0.080</td>
<td>0.956</td>
<td>2925</td>
</tr>
<tr>
<td></td>
<td>$\tau^2 = 1$</td>
<td>0.038</td>
<td>0.438</td>
<td>0.371</td>
<td>0.972</td>
<td>3497</td>
</tr>
</tbody>
</table>
**Compare with ICBayes**

**Table:** Simulation results based on 500 replicates of size $n = 500$ under the non-frailty PH model for pure interval-censored data. For each MCMC, we retain 10,000 scans without thinning after a burn-in period of 10,000 iterations.

<table>
<thead>
<tr>
<th>Method</th>
<th>Time</th>
<th>Parameter</th>
<th>BIAS</th>
<th>PSD</th>
<th>SD-Est</th>
<th>CP</th>
<th>Effective size</th>
</tr>
</thead>
<tbody>
<tr>
<td>spBayesSurv</td>
<td>4.14</td>
<td>$\beta_1 = 1$</td>
<td>0.032</td>
<td>0.136</td>
<td>0.132</td>
<td>0.962</td>
<td>1028</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\beta_2 = 1$</td>
<td>0.020</td>
<td>0.088</td>
<td>0.089</td>
<td>0.940</td>
<td>745</td>
</tr>
<tr>
<td>ICBayes</td>
<td>8.81</td>
<td>$\beta_1 = 1$</td>
<td>-0.022</td>
<td>0.133</td>
<td>0.124</td>
<td>0.958</td>
<td>342</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\beta_2 = 1$</td>
<td>-0.018</td>
<td>0.084</td>
<td>0.081</td>
<td>0.954</td>
<td>291</td>
</tr>
</tbody>
</table>
Variable Selection Results

- **Example 1:** \( x_{ij} = (x_{ij1}, \ldots, x_{ij5}) \) with \( x_{ij1} \sim \text{Bernoulli}(0.5) \) and
  \( x_{ij2}, \ldots, x_{ij5} \overset{iid}{\sim} \mathcal{N}(0, 1) \), and \( \beta = (1, 1, 0, 0, 0)' \).

- **Example 2:** \( x_{ij3} = x_{ij2} + 0.15z \) where \( z \sim \mathcal{N}(0, 1) \), yielding a 0.989 correlation between \( x_2 \) and \( x_3 \).

- **Example 3:** \( x_{ij} = (x_{ij1}, \ldots, x_{ij10}) \) with \( \beta = (1, 1, 1, 1, 1, 0, 0, 0, 0, 0)' \) and \( x_{ijk} | z \sim \mathcal{N}(z, 1) \) where \( z \sim \mathcal{N}(0, 1) \), which induces pairwise correlations of about 0.5.

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variables</td>
<td>Proportions</td>
<td>Variables</td>
</tr>
<tr>
<td>1 2</td>
<td>0.80</td>
<td>1 2</td>
</tr>
<tr>
<td>1 2 3</td>
<td>0.08</td>
<td>1 2 3</td>
</tr>
<tr>
<td>1 2 5</td>
<td>0.05</td>
<td>1 3</td>
</tr>
<tr>
<td>1 2 4</td>
<td>0.05</td>
<td>1 2 5</td>
</tr>
</tbody>
</table>