PALMETTO LECTURE IN STATISTICS, UNIVERSITY OF SOUTH CAROLINA

NETWORK RELIABILITY: A FRESH LOOK AT SOME BASIC QUESTIONS

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I. The Mathematical Representation of Communications Networks

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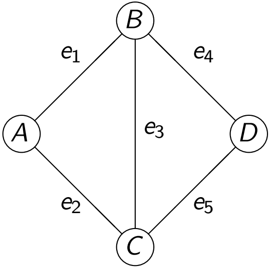

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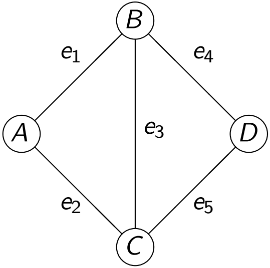

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For any integer $k = 2, \ldots, v$, *k-terminal connectivity* means that there is a working path linking any pair in a set of k distinguished vertices.

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(c) **An important special case.** Edge failure times have identical distributions, that is,

$$X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} F.$$

At time t_0, the probability that any given edge is working is $p = \overline{F}(t_0)$.

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- Assuming independence, if the common edge reliability p may be reasonably assumed to be a lower bound on all p_i, or if $\overline{F}_i(t) \geq \overline{F}(t)$ for all t, then the network's reliability is bounded below by the reliability of the network based on the i.i.d. assumption with $p_i \equiv p$ or $\overline{F}_i \equiv \overline{F}$.

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- The i.i.d. framework "levels the playing field" when comparing competing networks, and differences between network designs can be characterized through distribution-free summaries like "network signatures."

III. The Signature of a Network

Consider a network in the $G(v, n)$ class, and assume that its n edges have failure times $\{X_i\}$ which can be modeled as $X_1, X_2, \ldots, X_n \stackrel{\text{iid}}{\sim} F$. Suppose that a given type of connectivity is of interest. Let T be the time at which connectivity fails. The failure of connectivity coincides with a particular edge failure.

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Definition: The *signature* $\mathbf{s}$ of a $G(v, n)$ network is an n-dimensional probability vector whose ith element is $s_i = P(T = X_{i:n})$, where $X_{i:n}$ is the ith smallest X among the edge failure times $X_1, X_2, \ldots, X_n$.

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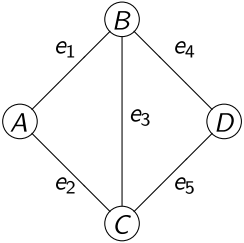

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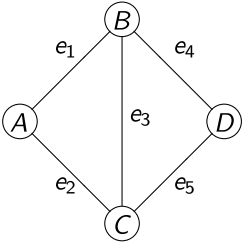

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- It follows that $P(T = X_{3:5}) = 3/5$.

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- Since it is not possible to connect 4 vertices with just 2 edges, connectivity will fail at or before the 3rd edge failure. Thus, $P(T = X_{4:5}) = 0$ and $P(T = X_{3:5}) = 4/5$.

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See Samaniego, F. J. (2007) *System Signatures and their Application in Engineering Reliability*, New York: Springer, for more details.

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- If network failure times T_1 and T_2 satisfy $T_1 <_{\text{sp}} T_2$, then $P(T_1 < T_2) > 0.5$, that is, the chances are that network 2 will last longer than network 1.

V. Comparing two $G(v, n)$ networks

When all edges work independently of each other and have a common probability p of working, the reliability of a network with n edges can be written as an nth degree polynomial. The reliability polynomial of the network can be expressed, in standard form, as

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Satyanarayana and Prabhakar (1978) provided an efficient technique for computing the "signed dominations" $\{d_r\}$ in the reliability polynomial.

The survival function of a network's lifetime T can be written as a function of **s** and F. At a fixed time t_0, where $P(X_j > t_0) = p$ for all j, this representation reduces to the reliability polynomial in "pq-form," where $q = 1 - p$:

$$\begin{aligned} h(p) &= \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} p^j q^{n-j} \\ &= \sum_{j=1}^{n} a_j \binom{n}{j} p^j q^{n-j} \ , \end{aligned}$$

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$$
P_{uv} = \begin{cases} 0 & \text{if } u + v \leq n \\ 1 & \text{if } u + v > n \end{cases} .
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$$(m^*_{i1}, \cdots, m^*_{ii}, 0, \cdots, 0) = \left(\underbrace{\frac{(i)_1}{(n)_1}, \frac{(i)_2}{(n)_2}, \cdots, \frac{(i)_i}{(n)_i}}_{i \text{ slots}}, \underbrace{0, \cdots, 0}_{n-i \text{ slots}} \right).$$

Expanding $q^{n-j} = (1-p)^{n-j}$ by the binomial theorem, we may identify each domination d_i as a linear combination of the elements $a_1, \ldots, a_n$. More specifically, we may write $\mathbf{d} = \mathbf{Ma}$, where

$$\mathbf{M} = \begin{pmatrix} \binom{n}{1}\binom{n-1}{0} & 0 & 0 & \cdots & 0 \\ -\binom{n}{1}\binom{n-1}{1} & \binom{n}{2}\binom{n-2}{0} & 0 & \cdots & 0 \\ \binom{n}{1}\binom{n-1}{2} & -\binom{n}{2}\binom{n-2}{1} & \binom{n}{3}\binom{n-3}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pm\binom{n}{1}\binom{n-1}{n-1} & \mp\binom{n}{2}\binom{n-2}{n-2} & \pm\binom{n}{3}\binom{n-3}{n-3} & \cdots & \binom{n}{n}\binom{n-n}{n-n} \end{pmatrix}$$

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Since $\mathbf{d} = \mathbf{MPs}$, we may express the relationship of interest to us as

$$\mathbf{s} = \mathbf{P}^{-1}\mathbf{M}^{-1}\mathbf{d}.$$

Theorem: (Boland, Samaniego and Vestrup, 2003) Let **d** and **s** denote the domination and signature vectors for a given network of order n. Then for $i = 1, \ldots, n$, we have

$$s_i = \sum_{j=1}^{n-i} \frac{(n-i+1)_j - (n-i)_j}{(n)_j} d_j + \frac{(n-i+1)_{n-i+1}}{(n)_{n-i+1}} d_{n-i+1} \ .$$

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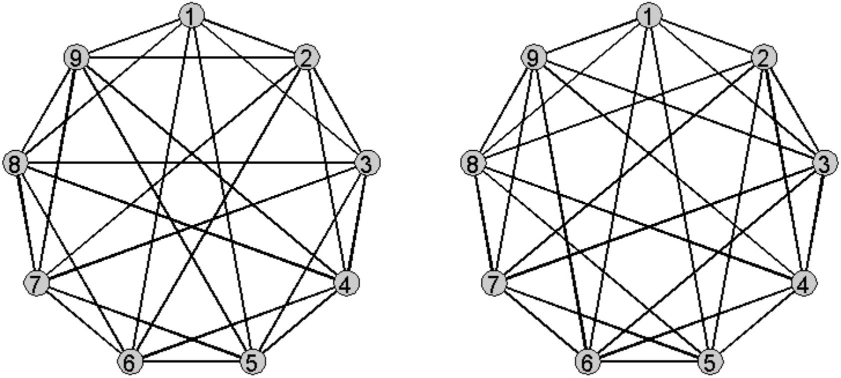
Having the relationship $\mathbf{s} = f(\mathbf{d})$ in hand enables us to exploit both the computational advantages of dominations and the interpretive value of signatures.

Example: Consider the comparison between the two $G(9, 27)$ networks pictured below.

Networks G_1 and G_2



The reliability polynomials are displayed below:

$$\begin{aligned}
\mathbf{h_{G_1}(p)} \quad = \quad & 419904p^{27} - 6021144p^{26} + 41705280p^{25} - 18489826p^{24} \\
& +586821717p^{23} - 1413876060p^{25} + 2677774329p^{21} \\
& -4074363810p^{20} + 5048856414p^{19} - 5135792742p^{18} \\
& +4303029693p^{17} - 2967712776p^{16} + 1676975886p^{15} \\
& -769265910p^{14} - 282176568p^{13} + 80853282p^{12} \\
& +17445456p^{11} - 2667060p^{10} + 257634p^{9} - 11828p^{8}
\end{aligned}$$

$$\begin{aligned}
\mathbf{h_{G_2}(p)} \quad = \quad & 414720p^{27} - 5934288p^{26} + 41015964p^{25} - 181453380p^{24} \\
& +574666025p^{23} - 1381692972p^{22} + 2611463517p^{21} \\
& -3965536554p^{20} - 4904464002p^{19} + 4979513718p^{18} \\
& +4164454729p^{17} - 2867022480p^{16} + 1617256842p^{15} \\
& -740601350p^{14} - 271201476p^{13} + 77576922p^{12} \\
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The difference $\mathbf{h_{G_1}(p)} - \mathbf{h_{G_2}(p)}$ is a 27th degree polynomial with alternating signs.

Table: Signature Tail Probabilities $S(x) = \sum_{i=x}^{27} s_i$ and Their Ratios

x	$S_{G_1}(x)$	$S_{G_2}(x)$	$S_{G_1}(x)/S_{G_2}(x)$
1	1.0	1.0	1.0
3	1.0	1.0	1.0
5	1.0	1.0	1.0
7	0.999970	0.999970	1.0
9	0.999149	0.999149	1.0
11	0.993612	0.993612	1.0
13	0.971744	0.971743	1.0000005
15	0.906907	0.906867	1.0000442
17	0.747317	0.746717	1.0008024
19	0.417560	0.415077	1.0059834
21	0.0	0.0	—
⋮	⋮	⋮	⋮
27	0.0	0.0	—

From the second and third columns of this table, we see that $\mathbf{s}_{G_1}(x) \geq_{\text{st}} \mathbf{s}_{G_2}(x)$ for all x, an inequality that immediately implies that $h_{G_1}(p) \geq h_{G_2}(p)$ for $p \in (0, 1)$. Further, $\mathbf{s}_{G_1} \geq_{\text{hr}} \mathbf{s}_{G_2}$, a conclusion that is not possible to obtain from an analysis of the polynomials $h_{G_1}(p)$ and $h_{G_2}(p)$ alone. This additional fact establishes that G_1 is not only better than the network G_2, it's actually better in quite a strong sense.

VI. The "Traditional" Approach to Identifying Uniformly Optimal Networks

The search for Uniformly Optimal Networks (UONs) among networks $G(v, n)$ of a given size includes work by Boesch, Li and Suffel (Networks, 1991), who, for example, identified the unique UON among networks in the $G(v, v-1)$, $G(v, v)$, $G(v, v+1)$ and $G(v, v+2)$ classes. The UON in the $G(v, v+3)$ class was later identified by Wang (1994).

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Letting $p = \overline{F}(t_0)$, where F is the *common lifetime distribution* of the network's edges, these investigators identified the network G^* (with lifetime T^*) which satisfies the inequality

$$P_{G^*}(T^* > t_0) = h_{G^*}(p) \geq h_G(p) = P_G(T > t_0) \quad \forall\, p \in [0, 1] \quad \text{and/or} \quad \forall\, t > t_0$$

for any network G (with lifetime T) in the class of interest. (That is, $T \leq_{\text{st}} T^*$.)

Impossibility Theorems: Myrvold, Cheung, Page and Perry (Networks, 1991) showed that for some classes of networks, e.g., the class $G\left(v,\ \binom{v}{2} - \frac{v}{2} - 1\right)$ for any even $v \geq 6$, *a UON does not exist*. They proved the existence of a network in this class which dominated every other network in the class for p sufficiently large, but was inferior to an alternative network if p is suitably small.

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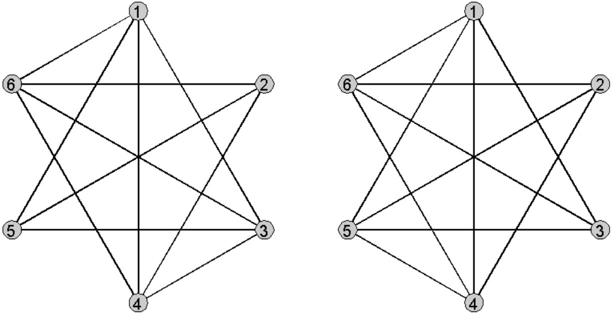
Networks G_8 and G_9



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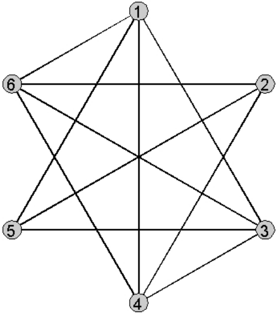

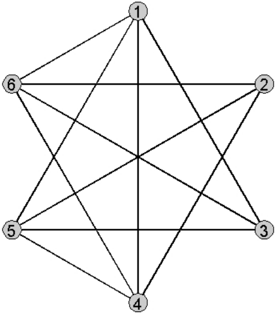

This paper all but squelched the vigorous research that had focused on the identification of UONs.

VII. Reversal of Fortune

Could it be that stochastic ordering is too strong a criterion to expect uniform optimality of a single member of a class $G(v, n)$? Consider the following empirical study of $G(6, 11)$ networks by McAssey and Samaniego.

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It is easy to verify that the $G(6, 11)$ class contains $\binom{15}{11} = 1365$ possible network designs. Suppose we are interested in all-terminal connectivity. When edges are independent with reliabilities p_i all equal to p, we can compute the signatures of each of these networks. One finds that there are precisely nine distinct signatures. These are shown in the following table.

Table: The Nine Possible Signatures of $G(6,11)$ Networks.

$$\mathbf{s}_1 = (0.0909, 0.0909, 0.0909, 0.1061, 0.1407, 0.2100, 0.2706, 0, 0, 0, 0)$$
$$\mathbf{s}_2 = (0.0000, 0.0182, 0.0485, 0.0939, 0.1662, 0.2835, 0.3896, 0, 0, 0, 0)$$
$$\mathbf{s}_3 = (0.0000, 0.0182, 0.0424, 0.0848, 0.1619, 0.2922, 0.4004, 0, 0, 0, 0)$$
$$\mathbf{s}_4 = (0.0000, 0.0182, 0.0364, 0.0758, 0.1489, 0.2879, 0.4329, 0, 0, 0, 0)$$
$$\mathbf{s}_5 = (0.0000, 0.0000, 0.0242, 0.0788, 0.1697, 0.3117, 0.4156, 0, 0, 0, 0)$$
$$\mathbf{s}_6 = (0.0000, 0.0000, 0.0182, 0.0636, 0.1541, 0.3117, 0.4524, 0, 0, 0, 0)$$
$$\mathbf{s}_7 = (0.0000, 0.0000, 0.0182, 0.0606, 0.1485, 0.3052, 0.4675, 0, 0, 0, 0)$$
$$\mathbf{s}_8 = (0.0000, 0.0000, 0.0121, 0.0515, 0.1398, 0.3095, 0.4870, 0, 0, 0, 0)$$
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Notes: Network G_8 has signature $\mathbf{s}_8$ and network G_9 has signature $\mathbf{s}_9$. The signatures $\mathbf{s}_8$ and $\mathbf{s}_9$ are not comparable relative to stochastic ordering. There are many $G(6, 11)$ networks that are "isomorphic" to a given network with any one of the signatures above.

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2. Further, the following preservation result holds for all $G(6,11)$ networks under stochastic precedence: if $\mathbf{s}_i \leq_{\text{sp}} \mathbf{s}_j$, then $T_i \leq_{\text{sp}} T_j$, where T_k represents the time of connectivity failure for network k, with $k = 1, \ldots, 9$. The following comparisons show that the network G_9 is the Uniformly Optimal Network relative to the stochastic precedence ordering:

$$P(T_9 > T_8) = 0.501, \quad P(T_9 > T_7) = 0.510, \quad P(T_9 > T_6) = 0.514$$
$$P(T_9 > T_5) = 0.528, \quad P(T_9 > T_4) = 0.534, \quad P(T_9 > T_3) = 0.546$$
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SO, THERE DOES EXIST A UNIFORMLY OPTIMAL NETWORK DESIGN AFTER ALL! IT'S CLEAR THAT THE CRITERION USED IN COMPARING NETWORKS MATTERS. BTW, THERE ARE 180 DISTINCT $G(6,11)$ NETWORKS THAT HAVE SIGNATURE $\mathbf{s}_9$.

VIII. Reliability Economics

Relative to the sp criterion, one is able to identify G_9 as the Universally Optimal Network within the $G(6, 11)$ class. This network, and those with the same signature, have the uniformly best performance among all $G(6, 11)$ networks.

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Now, suppose that network costs are taken into account. Consider the criterion function

$$m_r(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \sum_{i=1}^{n} a_i s_i \Bigg/ \left(\sum_{i=1}^{n} c_i s_i \right)^r ,$$

where the vectors **a** and **c** can be chosen arbitrarily within the context of two natural constraints: $0 < a_1 < \cdots < a_n$ and $0 < c_1 < \cdots < c_n$; the constant $r > 0$ is a calibration parameter that places more or less weight on costs depending on whether $r > 1$ or $r < 1$.

Again, taking

$$m_r(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \sum_{i=1}^{n} a_i s_i \Bigg/ \left(\sum_{i=1}^{n} c_i s_i \right)^r ,$$

we note that when $a_i = EX_{i:n}$, the numerator of m is simply ET. The linear form of the denominator of m arises, for example, in the "salvage model" for a wired network which yields an expected cost of the network equal to

$$EC = \sum_{i=1}^{n} (C_f + n(A - B) + Bi) s_i ,$$

where C_f is fixed cost of manufacturing the networks of interest, A is the cost of an individual edge and B is the salvage value of an edge that is used but working when the network fails.

Example 3. Suppose the lifetimes of edges in a $G(6, 11)$ network are i.i.d. exponential variables with mean life 100 hours. Taking $a_i = EX_{i:11}$ for $1 = 1, 2, \ldots, 11$, we may calculate the vector **a** as

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
9.1	19.1	30.2	42.7	57.0	73.7	93.7	118.7	152.0	202.0	302.0

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$$m(\mathbf{s}) = m_2(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \sum_{i=1}^{11} a_i s_i \bigg/ \left(\sum_{i=1}^{11} c_i s_i \right)^2 ,$$

and we obtain the following results for the criterion function m for the 9 distinct signatures of $G(6,11)$ networks:

$m(\mathbf{s}_1)$	$m(\mathbf{s}_2)$	$m(\mathbf{s}_3)$	$m(\mathbf{s}_4)$	$m(\mathbf{s}_5)$	$m(\mathbf{s}_6)$	$m(\mathbf{s}_7)$	$m(\mathbf{s}_8)$	$m(\mathbf{s}_9)$
5.034	4.757	4.748	4.740	4.710	4.698	4.697	4.687	4.686

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$$m(\mathbf{s}) = m_2(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \sum_{i=1}^{11} a_i s_i \Bigg/ \left(\sum_{i=1}^{11} c_i s_i \right)^2 ,$$

and we obtain the following results for the criterion function m for the 9 distinct signatures of $G(6,11)$ networks:

$m(\mathbf{s}_1)$	$m(\mathbf{s}_2)$	$m(\mathbf{s}_3)$	$m(\mathbf{s}_4)$	$m(\mathbf{s}_5)$	$m(\mathbf{s}_6)$	$m(\mathbf{s}_7)$	$m(\mathbf{s}_8)$	$m(\mathbf{s}_9)$
5.034	4.757	4.748	4.740	4.710	4.698	4.697	4.687	4.686

From this, we see that any $G(6,11)$ network with signature $\mathbf{s}_1$ is optimal on the basis of this performance vs. cost analysis, with $r = 2$.

Given the same criterion function with $c_i = U + V\,i$ and a_i as above, we varied r from 1 to 10, and for each r we varied both U and V independently from 1 to 100. At each pair (U, V) we evaluated the criterion function for each of the nine signatures, and noted which signature produces the maximum. For each r, the relative frequency distribution for the optimal signature over the grid of (U, V) pairs is shown below.

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When $r = 1$, signature 9 is optimal at all 10,000 pairs (U, V). For $r > 1$, optimality is distributed among signatures 1, 4, 8 and 9 over the grid, with signature 1 dominating:

Relative frequency of optimality

r	$\mathbf{s}_1$	$\mathbf{s}_4$	$\mathbf{s}_8$	$\mathbf{s}_9$
1	0.0000	0.0000	0.0000	1.0000
2	0.8718	0.0000	0.0353	0.0929
3	0.9450	0.0015	0.0118	0.0417
4	0.9657	0.0019	0.0069	0.0255
5	0.9761	0.0013	0.0046	0.0180
6	0.9820	0.0012	0.0033	0.0135
7	0.9859	0.0009	0.0032	0.0100
8	0.9887	0.0011	0.0019	0.0083
9	0.9906	0.0008	0.0019	0.0067
10	0.9921	0.0008	0.0021	0.0050

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3. What network designs offer scalable, secure, reliable communications under specific budgetary constraints? What are the advantages and disadvantages of modular designs? Can optimal designs be identified under various design constraints?